§12 Topological Spaces

Definition A topology on a set X is a collection $\mathscr{T}$ of subsets of X having the following properties:

(1) $\emptyset$ and X are in $\mathscr{T}$.

(2) The union of the elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

A set X for which a topology $\mathscr{T}$ has been specified is called a topological space.

Properly speaking, a topological space is an ordered pair $(X, \mathscr{T})$ consisting of a set X and a topology $\mathscr{T}$ on X, but we often omit specific mention of $\mathscr{T}$ if no confusion will arise.

Definition Let X be a topological space with topology $\mathscr{T}$. A subset U of X is called an open set of X if $U \in \mathscr{T}$, i.e. U belongs to the collection $\mathscr{T}$. Using this terminology, one can say that X is a topological space if

(1) $\emptyset$ and X are open.

(2) Arbitrary union of open sets is open.

(3) Any finite intersection of open sets is open.

Examples

1. Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X,some of which are indicated schematically in Figure 12.1. The diagram in the upper right-hand corner indicates the topology in which the open sets are X, $\emptyset, \{a, b\}, \{b\}$, and $\{b, c\}$. The topology in the upper left-hand corner contains only X and $\emptyset$, while the topology in the lower right-hand corner contains every subset of X. You can get other topologies on X by permuting a, b, and c.

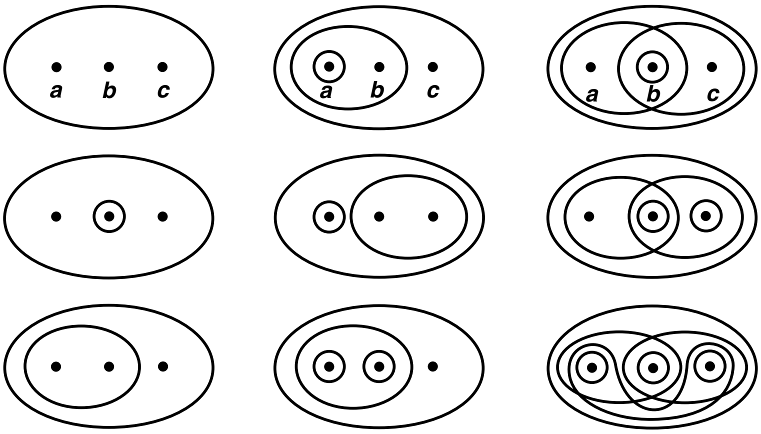


Figure 12.1

From this example, you can see that even a three-element set has many different topologies. But not every collection of subsets of X is a topology on X. Neither of the collections indicated in Figure 12.2 is a topology, for instance.

Figure 12.2

2. Let X be a set and let $\mathscr{T} = \mathscr{P}(X) = \{U \mid U \subset X\}$ be the collection of all subsets of X, called the power set of X. Then $\mathscr{T}$ is a topology on X and it is called the discrete topology on X.
3. Let X be a set and let $\mathscr{T} = \{\emptyset, X\}$ consist of $\emptyset$ and X only. Then $\mathscr{T}$ is a topology on X and it is called the trivial topology on X.
4. Let $X = \mathbb{R}$ and let
$$\mathscr{T}_f = \{U \mid X \setminus U \text{ is either finite subset or all of } X\}.$$
Then $\mathscr{T}_f$ is a topology on X, called the the finite complement topology.
5. Let $X = \mathbb{E}^n$ and let $\mathscr{T} = \{U \mid \forall x \in U, \exists \varepsilon = \varepsilon(x) > 0 \text{ s.t. } B_\varepsilon(x) \subset U\}$, where $B_\varepsilon(x) = \{y \in \mathbb{E}^n \mid d(x,y) < \varepsilon\}$ denotes the Euclidean ball with center x and radius ε. Then $\mathscr{T}$ is a topology on $\mathbb{E}^n$ and it is called the usual or standard topology on $\mathbb{E}^n$.

Definition Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set X. If $\mathscr{T}' \supset \mathscr{T}$, we say that $\mathscr{T}'$ is finer (or larger) than $\mathscr{T}$; if $\mathscr{T}'$ properly contains $\mathscr{T}$, we say that $\mathscr{T}'$ is strictly finer than $\mathscr{T}$. We also say that $\mathscr{T}'$ is coaser (or smaller) than $\mathscr{T}$, or strictly coarser,in these two respective situations. We say $\mathscr{T}$ is comparable with $\mathscr{T}'$ if either $\mathscr{T}' \supset \mathscr{T}$ or $\mathscr{T} \supset \mathscr{T}'$.

§13 Basis for a Topology

Definition If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing x $\Longrightarrow \bigcup_{B \in \mathscr{B}} B = X.$

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that
$$B_3 \subseteq B_1 \cap B_2$$
i.e. If B_1, $B_1 \in \mathscr{B}$ satisfy that $B_1 \cap B_2 \neq \emptyset$, then there is a $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Definition If $\mathscr{B}$ is a basis for a topology on X, then the topology $\mathscr{T}$ generated by $\mathscr{B}$ is defined by
$$\mathscr{T} = \{U \mid \forall x \in U, \exists B \in \mathscr{B} \text{ s.t. } x \in B \subseteq U\}$$
that is, U is an open subset of X (or $U \in \mathscr{T}$) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$.

Remark Note that

- if $B \in \mathscr{B} \implies B \in \mathscr{T}$, i.e. all basis elements are open in X under this definition, so that $\mathscr{B} \subseteq \mathscr{T}$.

- if $U \in \mathscr{T}$, then for each $x \in U$, there exists a $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$ and
$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq \bigcup_{x \in U} U = U \implies U = \bigcup_{x \in U} B_x$$
- $\emptyset \in \mathscr{T}$ since it satisfies the defining condition of openness vacuously.
- $X \in \mathscr{T}$ since for each $x \in X$ there is a basis element containing x and contained in X.
- if $\{U_\alpha\}_{\alpha \in J}$ is a collection of elements of $\mathscr{T}$ and if $U = \cup_{\alpha \in J} U_\alpha$, then $U \in \mathscr{T}$ since for each $x \in U$, there is an index α such that $x \in U_\alpha$ and since $U_\alpha \in \mathscr{T}$ there is a basis element B such that
$$x \in B \subseteq U_\alpha \implies x \in B \text{ and } B \subseteq U \implies U \in \mathscr{T}$$
- if U_1 and U_2 are elements of $\mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T}$ since for any $x \in U_1 \cap U_2$, there exist basis elements B_1, B_2 containing x such that $B_1 \subset U_1$ and $B_2 \subset U_2$. Also, by the second condition for a basis, there exists a basis element B_3 containing x such that $x \in B_3 \subset U_1 \cap U_2$ (see Figure 13.3) which implies that $U_1 \cap U_2 \in \mathscr{T}$, by definition.

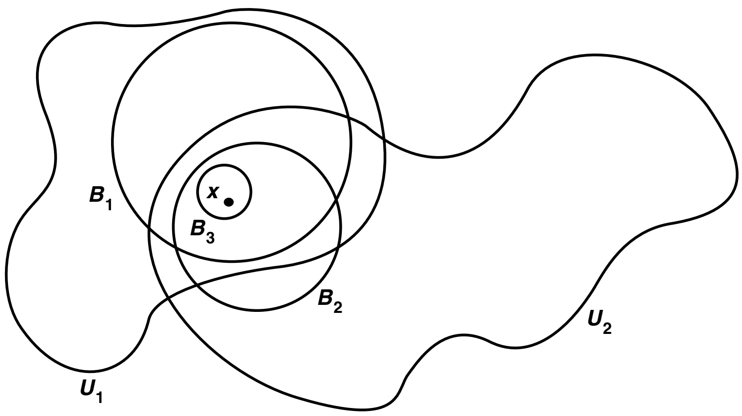


Figure 13.3

- if $U = U_1 \cap \cdots \cap U_n$ is any finite intersection of n elements of $\mathscr{T}$, then $U \in \mathscr{T}$ since this is trivial for the case when $n = 1$, and if we suppose this is true for the intersection of any $n-1$ elements then it is true for the intersection of any n elements since $U_1 \cap \cdots \cap U_{n-1} \in \mathscr{T}$ by hypothesis and
$$(U_1 \cap \cdots \cap U_n) = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n \in \mathscr{T}$$
by the result just proved.

Hence the collection $\mathscr{T}$ is indeed a topology on X.

Example 1. Let $\mathscr{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathscr{B}$ satisfies both conditions for a basis. The second condition is illustrated in Figure 13.1. In the topology generated by $\mathscr{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U.

Example 2. Let $\mathscr{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathscr{B}'$ satisfies both

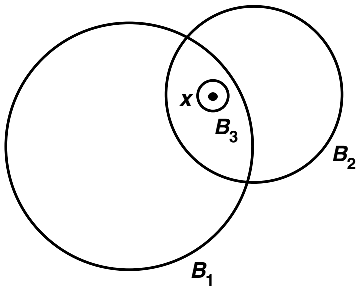


Figure 13.1

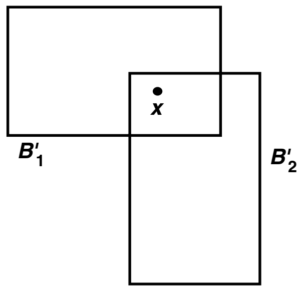


Figure 13.2

conditions for a basis. The second condition is illustrated in Figure 13.2; in this case, the condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty). As we shall see later, the basis $\mathscr{B}'$ generates the same topology on the plane as the basis $\mathscr{B}$ given in the preceding example.

Example 3. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

Another way of describing the topology generated by a basis is given in the following lemma:

Lemma 13.1 Let X be a set; let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. Then $\mathscr{T}$ equals the collection of all unions of elements of $\mathscr{B}$.

Proof Given any collection of elements of $\mathscr{B}$, they are also elements of $\mathscr{T}$, Because $\mathscr{T}$ is a topology, their union is in $\mathscr{T}$. Conversely, given $U \in \mathscr{T}$ choose for each $x \in U$ an element B_x of $\mathscr{B}$ such that $x \in B_x \subset U$. Then $U = \cup_{x\in U} B_x$, so U equals a union of elements of $\mathscr{B}$.

This lemma states that every open set U in X can be expressed as a union of basis elements. This expression for U is not, however, unique. Thus the use of the term "basis" in topology differs drastically from its use in linear algebra, where the equation expressing a given vector as a linear combination of basis vectors is unique.

We have described in two different ways how to go from a basis to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a basis generating it. Here is one way of obtaining a basis for a given topology; we shall use it frequently.

Lemma 13.2 Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.

Proof We must show that $\mathscr{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathscr{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathscr{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in $\mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let $\mathscr{T}$ be the collection of open sets of X; we must show that the topology $\mathscr{T}'$ generated by $\mathscr{C}$ equals the topology $\mathscr{T}$. First, note that if U belongs to $\mathscr{T}$ and if $x \in U$, then there is by hypothesis an element C of $\mathscr{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology $\mathscr{T}'$, by definition. Conversely, if W belongs to the topology $\mathscr{T}'$, then W equals a union of elements of $\mathscr{C}$, by the preceding lemma. Since each element of $\mathscr{C}$ belongs to $\mathscr{T}$ and $\mathscr{T}$ is a topology, W also belongs to $\mathscr{T}$.

When topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than another. One such criterion is the following:

Lemma 13.3 Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathscr{T}'$ is finer than $\mathscr{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.

Proof (2) $\implies$ (1). Given an element $U \in \mathscr{T}$, we wish to show that $U \in \mathscr{T}'$. For any $x \in U$, since $\mathscr{B}$ generates $\mathscr{T}$, there is an element $B \in \mathscr{B}$ such that $x \in B \subseteq U$. Condition (2) tells us there exists an element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. Then $x \in B' \subseteq U$, so $U \in \mathscr{T}'$, by definition.

(1) $\implies$ (2) For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, since $B \in \mathscr{T}$ by definition and $\mathscr{T} \subseteq \mathscr{T}'$ by condition (1), we have $B \in \mathscr{T}'$. Since $\mathscr{T}'$ is generated by $\mathscr{B}'$, there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.

Example 4. One can now see that the collection $\mathscr{B}$ of all circular regions in the plane generates the same topology as the collection $'$ of all rectangular regions; Figure 13.4 illustrates the proof. We shall treat this example more formally when we study metric spaces.

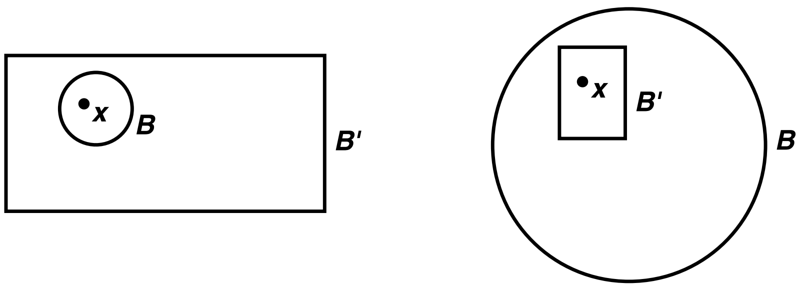


Figure 13.4

Definition A subbasis $\mathscr{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of all unions of finite intersections of elements of $\mathscr{S}$.

Note that if $\mathscr{B} = \{\bigcap_{i=1}^{m} S_i \mid S_i \in \mathscr{S},\ m \geq 1\}$ is the collection of all finite intersections of elements of $\mathscr{S}$, then $\mathscr{B}$ is a basis since $X = \bigcup_{S \in \mathscr{S}} S \subset \bigcup_{B \in \mathscr{B}} B \subset X \implies \bigcup_{B \in \mathscr{B}} B = X$ and

if $B_1 = S_1 \cap \cdots \cap S_m$, $B_2 = S_1' \cap \cdots \cap S_n' \in \mathscr{B} \implies B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n') \in \mathscr{B}$

Hence, by Lemma 13.1, the collection $\mathscr{T}$ of all unions of elements of $\mathscr{B}$ is a topology.

§16 The Subspace Topology

Definition Let X be a topological space with topology $\mathscr{T}$ and let Y be a subset of X. Then the collection

$$\mathscr{T}_Y = \{Y \cap U \mid U \in \mathscr{T}\}$$

is a topology on Y, called the subspace or induced topology. With this topology, Y is called a subspace of X; its open sets consists of all intersections of open sets of X with Y.

Since

$$\emptyset = Y \cap \emptyset \quad \text{and} \quad Y = Y \cap X \implies \emptyset, Y \in \mathscr{T}_Y,$$

$$\bigcap_{i=1}^{n} (U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y \in \mathscr{T}_Y, \quad \text{and} \quad \bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha\right) \cap Y \in \mathscr{T}_Y$$

$\mathscr{T}_Y$ is a topology on Y.

Lemma 16.1 If $\mathscr{B}$ is a basis for the topology of X then the collection

$$\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Proof Given U open in X and given $y \in U \cap Y$, we can choose an element B of $\mathscr{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 13.2 that $\mathscr{B}_Y$ is a basis for the subspace topology on Y.

Lemma 16.2 Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof Since U is open in Y, $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$.

Example Let $X = \mathbb{R}$ be the real line with the usual topology generated by the basis $\{(a, b) \mid a < b \in \mathbb{R}\}$ and let $Y = [0, 1]$. Then the basis for the subspace topology $\mathscr{T}_Y$ consists of the following:

$$(a, b) \cap Y = \begin{cases} (a, b) & \text{if } 0 < a < b < 1 \\ [0, b) & \text{if } a < 0 < b < 1 \\ (a, 1] & \text{if } 0 < a < 1 < b \\ \emptyset & \text{if } b < 0 \text{ or } 1 < a \\ [0, 1] & \text{if } a < 0 < 1 < b \end{cases}$$

Example Let $X = \mathbb{R}$ be the real line with the usual topology as above, and let $Y = [0, 1) \cup \{2\}$. Note that the one-point set $\{2\}$ and $[0, 1)$ are open in the subspace topology $\mathscr{T}_Y$.

§17 Closed Sets and Limit Points

Definition Let X be a topological space. A subset F of X is called an closed set of X if $F^c = X \setminus F \in \mathscr{T}$, i.e. the complement subset of F in X is an open set of X.

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X:

Theorem 17.1 Let X be a topological space. Then the following conditions hold:

(1) $\emptyset$ and X are closed.
(2) Arbitrary intersection of closed sets is closed.
(3) Any finite union of closed sets is closed.

Proof (1) $\emptyset$ and X are closed because they are the complements of the open sets X and $\emptyset$, respectively.

(2) Given a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, we apply DeMorgan's law,

$$X \setminus \bigcap_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J} (X \setminus A_\alpha)\,.$$

Since the sets $X \setminus A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\cap_{\alpha\in J} A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, \ldots, n$, consider the equation

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i)\,.$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\cup_{i=1}^{n} A_i$ is closed.

Theorem 17.2 Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof Assume that $A = C \cap Y$, where C is closed in X. (See Figure 17.1.) Then $X \setminus C$ is open in X, so that $(X \setminus C) \cap Y$ is open in Y, by definition of the subspace topology. But$(X \setminus C) \cap Y = Y \setminus A$ Hence $Y \setminus A$ is open in Y, so that A is closed in Y.

Conversely, assume that A is closed in Y. (See Figure 17.2.) Then $Y \setminus A$ is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set $X \setminus U$ is closed in X, and $A = Y \cap (X \setminus U)$ so that A equals the intersection of a closed set of X with Y, as desired.

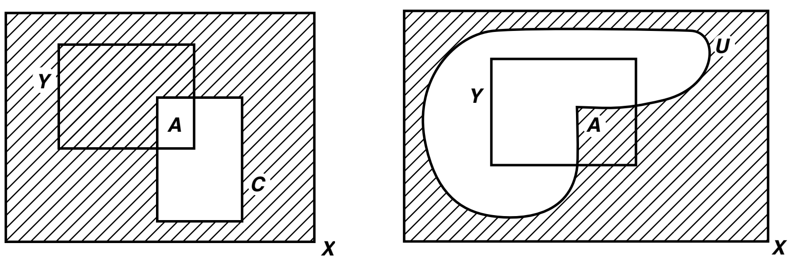


Figure 17.1 *Figure 17.2*

A set A that is closed in the subspace Y may or may not be closed in the larger space X. As was the case with open sets, there is a criterion for A to be closed in X as follows.

Theorem 17.3 Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Definition Let X be a topological space. A subset U is called a open neighborhood of p if U is an open set containing p. Note that if U is an open subset of X, then it is a neighborhood of each point $p \in U$.

Definition Let A be a subset of a topological space X. A point p of X is called a limit point (or accumulation point) of A if every open neighborhood U of p contains at least one point of $A \setminus \{p\}$, i.e.

$$U \cap A \setminus \{p\} \neq \emptyset.$$

Let A' denote the set of limit points of A. Note that a limit point of A may not be a point in A.

Examples

1. Let $X = \mathbb{R}$ and let $A = \{1/n \mid n \in \mathbb{N}\}$. Then A has exactly one limit point, namely the origin.
2. Let $X = \mathbb{R}$ and let $A = [0, 1)$. Then $[0, 1]$ is the set of limit points of A.
3. Let $X = \mathbb{E}^3$ and let $A = \{(x, y, z) \mid x,\ y,\ z \in \mathbb{Q}\}$. Then $\mathbb{E}^3$ is the set of limit points of A.
4. Let $X = \mathbb{E}^3$ and let $A = \{(x, y, z) \mid x,\ y,\ z \in \mathbb{Z}\}$. Then A does not have any limit points.
5. Let $X = \mathbb{R}$ with the finite complement topology $\mathscr{T}_f$. If we take A to be an infinite subset of X, then every point of X is a limit point of A. On the other hand a finite subset of X has no limit points in this topology.

Corollary 17.7 A set is closed if and only if it contains all of its limit points.

Proof If A is closed, then $X \setminus A$ is open. Since

$$A \cap (X \setminus A) \setminus \{p\} = \emptyset \quad \forall\, p \in X \setminus A,$$

$X \setminus A$ does not contain any limit point of A. Therefore A contains all of its limit points.

Conversely, suppose A contains all of its limit points and let $p \in X \setminus A$. Since p is not a limit point of A, there is a neighborhood U of p such that

$$U \cap A = \emptyset \implies p \in U \subset X \setminus A$$

This implies that $X \setminus A$ is a neighborhood of each of its points and consequently open. Therefore A is closed.

Definition Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A, and the closure of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $\operatorname{Int} A$ and the closure of A is denoted by $\operatorname{Cl} A$ or by $\bar{A}$. Obviously $\operatorname{Int} A$ is an open set and $\bar{A}$ is a closed set; furthermore,

$$\operatorname{Int} A \subset A \subset \bar{A}.$$

If A is open, $A = \operatorname{Int} A$; while if A is closed, $A = \bar{A}$, that is, A is closed if and only if it is equal to its closure.

Theorem 17.4 If Y is a subspace of X, A is a subset of Y and $\bar{A}$ denotes the closure of A in X, then the closure of A in Y equals $\bar{A} \cap Y$.

Proof Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A}\cap Y$ is closed in Y by Theorem 17.2. Since $\bar{A}\cap Y$ contains A, and since by definition B equals the intersection of all closed subsets of Y containing A, we must have $B\subset \bar{A}\cap Y$.

On the other hand, we know that B is closed in Y. Hence by Theorem 17.2, $B=C\cap Y$ for some set C closed in X. Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A}\subset C$. Then $\left(\bar{A}\cap Y\right)\subset (C\cap Y)=B$.

Theorem 17.5 Let A be a subset of the topological space X.

(a) Then $x\in\bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x\in\bar{A}$ if and only if every basis element B containing x intersects A.

Proof Consider the statement in (a). It is a statement of the form $P \iff Q$. Let us transform each implication to its contrapositive, thereby obtaining the logically equivalent statement $(\text{not}\, P) \iff (\text{not}\, Q)$. Written out, it is the following:

$$x\notin\bar{A} \iff \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A.$$

In this form, our theorem is easy to prove. If x is not in $\bar{A}$, the set $U=X\setminus\bar{A}$ is an open set containing x that does not intersect A, as desired. Conversely, if there exists an open set U containing x which does not intersect A, then $X\setminus U$ is a closed set containing A. By definition of the closure $\bar{A}$, the set $X\setminus U$ must contain $\bar{A}$; therefore, x cannot be in $\bar{A}$.

Statement (b) follows readily. If every open set containing x intersects A, so does every basis element B containing x, because B is an open set. Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

Theorem 17.6 Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\bar{A}=A\cup A'.$$

Proof If x is in A', every neighborhood of x intersects A (in a point different from x). Therefore, by Theorem 17.5, x belongs to $\bar{A}$. Hence $A'\subset\bar{A}$. Since by definition $A\subset\bar{A}$, it follows that $A\cup A'\subset\bar{A}$.

To demonstrate the reverse inclusion, we let x be a point of $\bar{A}$ and show that $x\in A\cup A'$. If x happens to lie in A, it is trivial that $x\in A\cup A'$; suppose that x does not lie in A. Since $x\in\bar{A}$, we know that every neighborhood U of x intersects A; because $x\notin A$ the set U must intersect A in a point different from x. Then $x\in A'$ so that $x\in A\cup A'$ as desired.

Definition Let A be a subset of a topological space X. The frontier (or boundary) of A, usually denoted ∂A, is the intersection of the closure of A with the closure of $X\setminus A$, i.e.

$$\partial A=\bar{A}\cap\overline{X\setminus A}$$

Definition A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 17.8 Every finite point set in a Hausdorff space X is closed.

Proof It suffices to show that every one-point set $\{x_0\}$ is closed. If x s a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V, respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed.

Definition In an arbitrary topological space, one says that a sequence $x_1, x_2, \ldots$ of points of the space X converges to the point x of X provided that, corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Theorem 17.10 If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y.

§18 Continuous Functions

Definition Let X and Y be topological spaces. A function $f : X \to Y$ is continuous on X if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Remark Let us note that if the topology of the range space Y is given by a basis $\mathscr{B}$, then to prove continuity of f it suffices to show that the inverse image of every basis element is open: The arbitrary open set V of Y can be written as a union of basis elements

$$V = \bigcup_{\alpha \in J} B_\alpha.$$

Then

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha),$$

so that $f^{-1}(V)$ is open if each set $f^{-1}(B_\alpha)$ is open.

If the topology on Y is given by a subbasis $\mathscr{S}$, to prove continuity of f it will even suffice to show that the inverse image of each subbasis element is open: The arbitrary basis element B for Y can be written as a finite intersection $S_1 \cap \cdots \cap S_n$ of subbasis elements; it follows from the equation

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$$

that the inverse image of every basis element is open.

Definition A function $h : X \to Y$ is called a homeomorphism if it is one-to-one, onto, continuous and has a continuous inverse $h^{-1} : Y \to X$. When such a function exists, X and Y are called homeomorphic (or topologically equivalent) spaces.

The map $f : X \to Y$ is called a topological imbedding, or simply an imbedding, of X in Y if $f : X \to Y$ is an injective continuous map from X into Y.

Example Let the real line $\mathbb{R}$ and the complex plane $\mathbb{C} \equiv \mathbb{R}^2$ be given the usual topologies, let $X = [0,1) \subset \mathbb{R}$ and $Y = C = \{z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$ be given respectively the subspace topologies, and let $f : [0,1) \to C$ be defined by

$$f(t) = e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t \quad \text{for each } t \in [0,1) \implies f \text{ is one-to-one and onto.}$$

Note that for each open disk $B_r(p) = \{z \in \mathbb{C} \mid \|z - p\| < r\}$ in $\mathbb{C}$, since $B_r(p) \cap C$ is open in C and $f^{-1}(B_r(p) \cap C)$ is open in $[0,1)$, $f : [0,1) \to C$ is continuous. However, its inverse $f^{-1} : C \to [0,1)$ is not continuous since for example $\left(f^{-1}\right)^{-1}([0,1/2)) = \{p \in C \mid f^{-1}(p) \in [0,1)\}$ is not open in (the subspace) C while $[0,1/2)$ is open in (the subspace) $[0,1)$).

Theorem Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composition $g \circ f : X \to Z$ is a continuous function.

Proof Let O be an open set in Z. Since

$$(g \circ f)^{-1}(O) = f^{-1}\left(g^{-1}(O)\right)$$

and $g^{-1}(O)$ is open in Y because g is continuous, so $f^{-1}\left(g^{-1}(O)\right)$ must be open in X by the continuity of f. Therefore $g \circ f : X \to Z$ is continuous.

Theorem Suppose $f : X \to Y$ is continuous, and let $A \subseteq X$ have the subspace topology. Then the restriction $f|_A : A \to Y$ is continuous.

Proof Let O be an open set in Y and notice that

$$(f|_A)^{-1}(O) = A \cap f^{-1}(O).$$

Since f is continuous, $f^{-1}(O)$ is open in X. Therefore $(f|_A)^{-1}(O)$ is open in the subspace topology on A, and the continuity of $f|_A$ follows from the preceding Theorem.

Definition The map $1_X : X \to X$, defined by $1_X(x) = x$ for each $x \in X$, is called the identity map of X. If we restrict 1_X to a subspace A of X we obtain the inclusion map $i : A \to X$.

Theorem 18.1 Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(1) $f : X \to Y$ is continuous.

(2) If β is a base for the topology of Y, the inverse image of every member of β is open in X.

(3) For every subset A of X, one has $f(\bar{A}) \subseteq \overline{f(A)}$.

(4) For every subset B of Y, the set $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$

(5) The inverse image of each closed set in Y is closed in X.

Proof

[(1) ⇒ (2)] For each $B \in \beta$, since B is an open set in the topology generated by β, $f^{-1}(B)$ is open in X.

[(2) ⇒ (3)] Let A be a subset of X. Since $\bar{A} = A \cup A'$ and $f(A) \subseteq \overline{f(A)} = f(A) \cup f(A)'$, it suffices to show that if $f(x) \in f(\bar{A})$, $x \in \bar{A} \setminus A$ and if $f(x) \notin f(A)$, then $f(x) \in f(A)'$.

Suppose that $f(x) \in f(\bar{A})$, $x \in \bar{A} \setminus A$, $f(x) \notin f(A)$ and N is an open neighborhood of $f(x)$ in Y. Since β is a base for the topology of Y, there exists a basis element (an open subset) B in β such that

$$f(x) \in B \subseteq N \implies x \in f^{-1}(B) \subseteq f^{-1}(N).$$

Assuming (2), the set $f^{-1}(B)$ is open in X and is therefore an open neighborhood of x. Also since

$$x \in A' \implies f^{-1}(B) \cap A \neq \emptyset \implies B \cap f(A) \neq \emptyset$$

and since

$$B \cap f(A) \subseteq N \cap f(A) \implies N \cap f(A) \setminus \{f(x)\} = N \cap f(A) \neq \emptyset \implies f(x) \in f(A)' \subset \overline{f(A)}$$

This completes the proof of (3).

[(3) ⇒ (4)] For any subset B of Y, since $f^{-1}(B)$ is a subset of X and by assuming (3), we have

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f(f^{-1}(B))} \subseteq \bar{B} \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$$

[(4) ⇒ (5)] If B is a closed subset of Y, since $\bar{B} = B$ and by assuming (4), we have

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B}) = f^{-1}(B) \subseteq \overline{f^{-1}(B)} \implies f^{-1}(B) = \overline{f^{-1}(B)}$$

and thus $f^{-1}(B)$ is closed in X.

[(5) ⇒ (1)] For each open set O of Y, since

$$X \setminus f^{-1}(O) = \{x \in X \mid f(x) \notin O\} = \{x \in X \mid f(x) \in Y \setminus O\} = f^{-1}(Y \setminus O),$$

$Y \setminus O$ is closed in Y and by assuming (5), we have $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$ is closed in X and thus $f^{-1}(O)$ is open in X. This shows that $f : X \to Y$ is continuous.

§15 The Product Topology on $X \times Y$

Definition Let X and Y be topological spaces. The product topology on $X \times Y$ is defined to be the topology generated by the basis

$$\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Since

- $X \times Y \in \mathscr{B} \implies \bigcup_{U \times V \in \mathscr{B}} U \times V = X \times Y,$
- if $U_1 \times V_1,\ U_2 \times V_2 \in \mathscr{B}$, then $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively, and

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathscr{B} \quad \text{as in Figure 15.1,}$$

$\mathscr{B}$ is a basis.

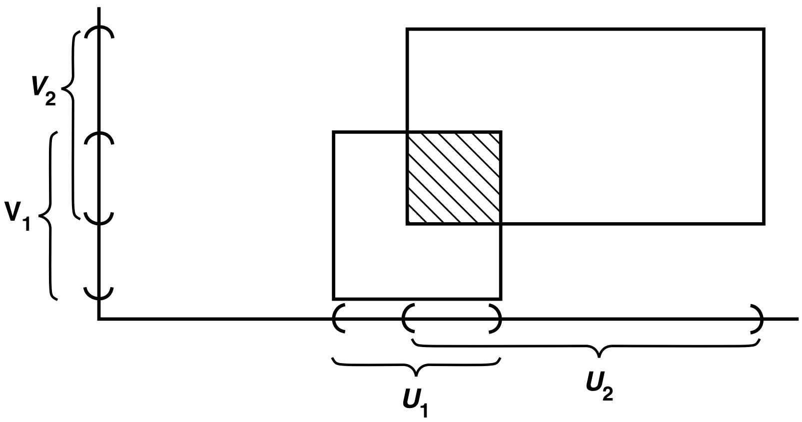


Figure 15.1

The set $X \times Y$, when equipped with the product topology, is called a product space.

In general, if $X_1, X_2, \ldots, X_n$ are topological spaces, the product topology on $X_1 \times X_2 \times \cdots \times X_n$ is the topology generated by the base $\mathscr{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \text{ is open in } X_i,\ 1 \le i \le n\}$.

For each $1 \le i \le n$, the function $\pi_i : X_1 \times \cdots \times X_i \times \cdots \times X_n \to X_i$ defined by $\pi_i(x_1, \cdots, x_i, \cdots, x_n) = x_i$ is called the projection of $X_1 \times \cdots \times X_n$ onto its i^{th} factor. Note that for each open subset U_i of X_i, since $\pi_i^{-1}(U_i) = X_1 \times \cdots \times U_i \times \cdots \times X_n$ is open in $X_1 \times \cdots \times X_n$, the projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is continuous .

Theorem 15.2 The collection

$$\mathscr{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

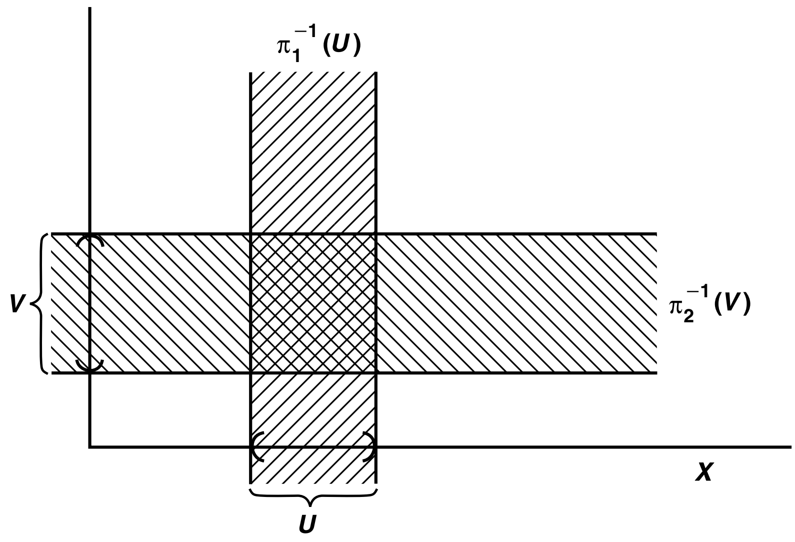


Figure 15.2

Proof Let $\mathscr{T}$ denote the product topology on $X \times Y$; let $\mathscr{T}'$ be the topology generated by $\mathscr{S}$. Because every element of $\mathscr{S}$ belongs to $\mathscr{T}$, so do arbitrary unions of finite intersections of elements of $\mathscr{S}$. Thus $\mathscr{T}' \subset \mathscr{T}$.

On the other hand, every basis element $U \times V$ for the topology $\mathscr{T}$ is a finite intersection of elements of $\mathscr{S}$ since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Therefore, $U \times V$ belongs to $\mathscr{T}'$, so that $\mathscr{T} \subset \mathscr{T}'$ as well.

Theorem Let X, Y and Z be topological spaces and $f : Z \to X \times Y$ be a function from Z into $X \times Y$. Then $f : Z \to X \times Y$ is continuous if and only if the two composite functions (coordinate functions) $\pi_1 \circ f : Z \to X$, $\pi_2 \circ f : Z \to Y$ are both continuous.

Proof ($\Longrightarrow$) If $f : Z \to X \times Y$ is continuous, then $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, by the continuity of the projections π_1, π_2.

($\Longleftarrow$) If both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, then $f : Z \to X \times Y$ is continuous since for each basic open set $U \times V$ of $X \times Y$,

$$f^{-1}(U \times V) = (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V) \quad \text{is open in } Z.$$

Theorem The product space $X \times Y$ is a Hausdorff space if and only if both X and Y are Hausdorff.

Proof ($\Longrightarrow$) Suppose that $X \times Y$ is Hausdorff. Given distinct points $x_1, x_2 \in X$, we choose a point $y \in Y$ and find disjoint basic open sets $U_1 \times V_1$, $U_2 \times V_2$ in $X \times Y$ such that $(x_1, y) \in U_1 \times V_1$ and $(x_2, y) \in U_2 \times V_2$.

Then U_1, U_2 are disjoint open neighborhoods of x_1 and x_2 in X. Therefore X is a Hausdorff space.

The argument for Y is similar.

($\Longleftarrow$) Suppose that X and Y are both Hausdorff spaces. Let (x_1, y_1) and (x_2, y_2) be distinct points of $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both).

If $x_1 \neq x_2$, since X is Hausdorff, there are disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Since $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$ and $(U_1 \times Y) \cap (U_2 \times Y) = \emptyset$, $X \times Y$ is a Hausdorff space.

The argument for $y_1 \neq y_2$ is similar.

§20 The Metric Topology

Definition A metric (or distance) function on a set X is a real-valued function $d : X \times X \to \mathbb{R}$ defined on the Cartesian product $X \times X$ such that for all x, y, $z \in X$:

(a) $d(x, y) \geq 0$ and equality holds if and only if $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) + d(y, z) \geq d(x, z)$.

Given a metric d on X and a positive number $\varepsilon > 0$, the set $B_d(x, \varepsilon)$ defined by

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$$

is called the ε-ball centered at x.

Definition If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the metric topology induced by d, and the set X together with metric d, usually denoted (X, d), is called a metric space.

The first condition for a basis is trivial, since $B(x, \varepsilon)$ for any $\varepsilon > 0$. Before checking the second condition for a basis, we show that if y is a point of the basis element $B(x, \varepsilon)$, then there is a basis element $B(y, \delta)$ centered at y that is contained in $B(x, \varepsilon)$. Define δ to be the positive number $\varepsilon - d(x, y)$. Then$B(y, \delta) \subset B(x, \varepsilon)$ for if $z \in B(y, \delta)$ then $d(y, z) < \varepsilon - d(x, y)$ from which we conclude that

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon$$

Now to check the second condition for a basis, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Letting δ be the smaller of δ_1 and δ_2, we conclude that $B(y, \delta) \subset B_1 \cap B_2$.

Using what we have just proved, we can rephrase the definition of the metric topology as follows:

A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

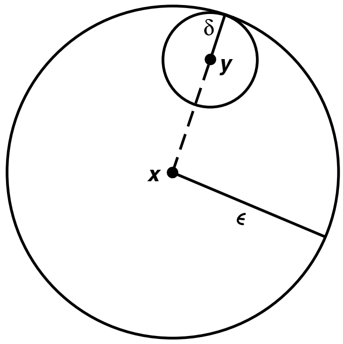


Figure 20.1

Clearly this condition implies that U is open. Conversely, if U is open, it contains a basis element $B = B_d(x, \varepsilon)$ containing y, and B in turn contains a basis element$B_d(y, \delta)$ centered at y.

Definition If X is a topological space, X is said to be metrizable if there exists a metric d on the set X that induces the topology of X. A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

Definition Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the norm of x by the equation

$$\|x\| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2};$$

and we define the euclidean metric (or usual metric, standard metric) d on $\mathbb{R}^n$ by the equation

$$d(x, y) = \|x - y\| = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{1/2}.$$

We define the square metric ρ by the equation

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

The proof that d is a metric requires some work; it is probably already familiar to you. If not, a proof is outlined in the exercises. We shall seldom have occasion to use this metric on $\mathbb{R}^n$.

To show that ρ is a metric is easier. Only the triangle inequality is nontrivial. From the triangle inequality for $\mathbb{R}$ it follows that for each positive integer i,

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

Then by definition of ρ,

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq \rho(x, y) + \rho(y, z).$$

As a result

$$\rho(x, y) \leq \max_{1 \leq i \leq n} |x_i - z_i| \leq \rho(x, y) + \rho(y, z),$$

as desired

Lemma 20.2 Let d and d' be two metrics on the set X; let $\mathscr{T}$ and $\mathscr{T}'$ be the topologies they induce, respectively. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$$

Proof Suppose that $\mathscr{T}'$ is finer than $\mathscr{T}$. Given the basis element $B_d(x, \varepsilon)$ for $\mathscr{T}$, there is by Lemma 13.3 a basis element B' for the topology $\mathscr{T}'$ such that $x \in B' \in B_d(x, \varepsilon)$. Within B' we can find a ball $B_{d'}(x, \delta)$ centered at x.

Conversely, suppose the $\delta - \varepsilon$ condition holds. Given a basis element B for $\mathscr{T}$ containing x, we can find within B a ball $B_d(x, \varepsilon)$ centered at x. By the given condition, there is a δ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$. Then Lemma 13.3 applies to show $\mathscr{T}'$ is finer than $\mathscr{T}$.

Theorem 20.3 The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

§22 The Quotient Topology

Definition Let X and Y be topological spaces; let $p : X \to Y$ be a surjective function. The map p is said to be a quotient map, provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Remark

- This condition is stronger than continuity.
- An equivalent condition is to require that a subset A of Y is closed in Y if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follows from the equation

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

- A subset C of X is saturated (with respect to the surjective continuous function $p : X \to Y$) if C contains every set $p^{-1}(y)$ that it intersects. Thus a subset C of X is saturated if $C = p^{-1}(p(C))$.

  So, p is a quotient map if p is continuous and p maps saturated open sets of X to open sets of Y (or saturated closed sets of X to open sets of Y).
- A map $f : X \to Y$ is said to be an open map if for each open set U of X, the set $f(U)$ is open in Y. It is said to be an closed map if for each closed set A of X, the set $f(A)$ is closed in Y.

  It follows immediately from the definition that if $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

Definition If X is a space and A is a set and if $p : X \to A$ is a surjective function, then there exists exactly one topology $\mathscr{T}$ on A relative to which p is a quotient map; it is called the quotient topology induced by p.

Remark Note that $\mathscr{T}$ is defined by

$$\mathscr{T} = \{U \subseteq A \mid p^{-1}(U) \text{ is open in } X\},$$

and $\mathscr{T}$ is a topology since

$$\begin{aligned}
&p^{-1}(\emptyset) = \emptyset \text{ and } p^{-1}(A) = X \implies \emptyset, A \in \mathscr{T}, \\
&p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha) \implies \text{ if } U_\alpha \in \mathscr{T}, \forall \alpha \in J \text{ then } \bigcup_{\alpha \in J} U_\alpha \in \mathscr{T}, \\
&p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i) \implies \text{ if } U_i \in \mathscr{T}, \forall 1 \leq i \leq n \text{ then } \bigcap_{i=1}^{n} U_i \in \mathscr{T}.
\end{aligned}$$

Theorem Let $f : X \to Y$ be an onto continuous function. If f maps open sets of X to open sets of Y, or closed sets to closed sets, then f is a quotient map.

Proof Suppose f maps open sets to open sets.

If U is open in Y, since f is continuous, $f^{-1}(U)$ is open in X.

Conversely, if $f^{-1}(U)$ is open in X for some subset U of Y, since f is onto and f is an open mapping, $U = f(f^{-1}(U))$ is open in Y. Hence f is a quotient map.

Corollary Let $f : X \to Y$ be an onto continuous function. If X is compact and Y is Hausdorff, then f is a quotient map.

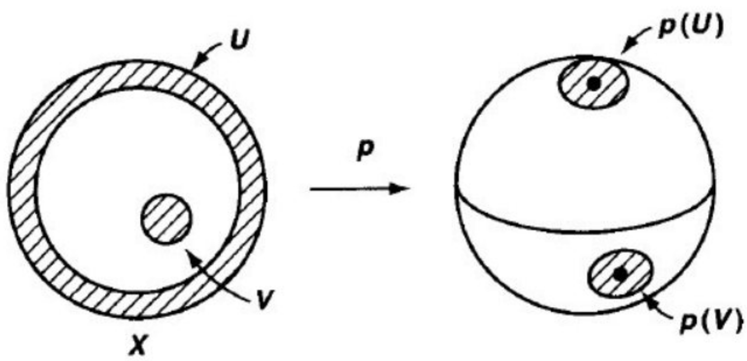
Proof Let C be a closed subset of X. Since X is compact, $f : X \to Y$ is continuous and Y is Hausdorff, $f(C)$ is a compact subset and hence a closed subset of Y. This implies that f takes closed sets to closed sets. Hence f is a quotient map.

Examples

1. Let $\pi_1 : X \times Y \to X$ be the projection map; π_1 is continuous and surjective. If $U \times V$ is a basis element for $X \times Y$, the image set $\pi_1(U \times V) = U$ is open in X. It follows readily that π_1 is an open map. In general, π_1 is not a closed map; the projection $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ carries the closed set $\{(x, y) \mid xy = 1\}$ onto the the nonclosed set $\mathbb{R} \setminus \{0\}$, for instance.
2. Let X be the subspace $[0, 1] \cup [2, 3]$ of $\mathbb{R}$, and let Y be the subspace $[0, 2]$ of $\mathbb{R}$. The map $p : X \to Y$ defined by
$$p(x) = \begin{cases} x & \text{for } x \in [0, 1], \\ x - 1 & \text{for } x \in [2, 3] \end{cases}$$
is readily seen to be surjective, continuous, closed map. Since $(1/2,\ 1]$ is open in X, $p((1/2,\ 1]) = (1/2,\ 1]$ is not open Y, $p : X \to Y$ is not an open map.
3. Let $A = \{a,\ b,\ c\}$ be a set of three points and let $p : \mathbb{R} \to A$ be defined by
$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0 \end{cases}$$
Then the quotient topology on A induced by π is $\mathscr{T} = \{\{a\},\ \{b\},\ \{a,\ b\},\ \{a,\ b,\ c\},\ \emptyset\}$, p is an open map and it is not a closed map.

Definition Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X.

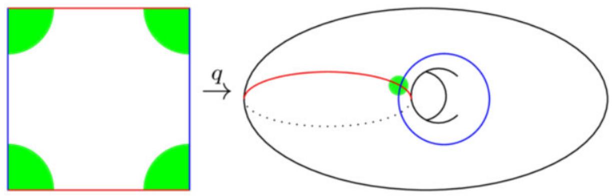
Example 4 Let X be the closed unit ball $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$, and let X^* be the partition of X defined as follows:
$$X^* = \bigcup_{x^2+y^2<1} \{(x, y)\} \cup \{(x, y) \mid x^2 + y^2 = 1\}.$$
Note that X^* consists of all the disjoint one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$ and the circle subset $\{(x, y) \mid x^2 + y^2 = 1\}$ of X. Typical open sets in X of the form $p^{-1}(U)$ are pictured by the shaded regions in the following figure. One can show that X^* is homeomorphic with the unit 2-sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$.

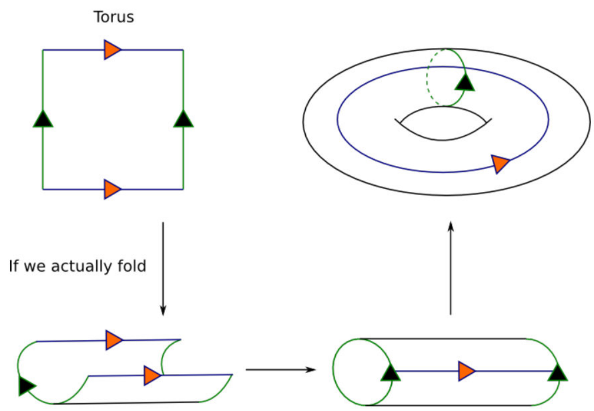


Example 5 Let X be the rectangle $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Define a partition X^* of X as follows:

$$X^* = \bigcup_{0<x,\,y<1}\{(x,y)\}\cup\bigcup_{0<x<1}\{(x,0),\,(x,1)\}\cup\bigcup_{0<y<1}\{(0,y),\,(1,y)\}\cup\{(0,0),\,(0,1),\,(1,0),\,(1,1)\}.$$

There are 4 kinds of typical open sets in X of the form $p^{-1}(U)$ and one of them is shown in the following figure.



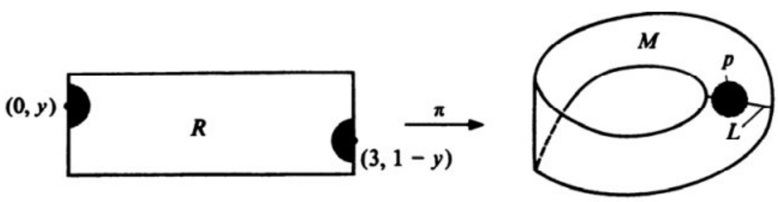


This description of X^* is just a mathematical way of pasting the edges of a rectangle together to form a torus.

Example 6 Let R be the rectangle $[0,3]\times[0,1]$ in $\mathbb{R}^2$. Define a partition R^* of R as follows:

$$R^* = \bigcup_{0<x<3,\,0\le y\le 1}\{(x,y)\}\cup\bigcup_{0\le y\le 1}\{(0,y),\,(3,1-y)\}.$$

Identify the subsets of R^* with the points of our Möbius strip M, and define the map $\pi : R\to M$ by sending each point of R to the subset of the partition in which it lies.



Note that the union of two half discs in R, centers $(0,y)$, $(3,1-y)$ and of equal radius, maps via π to an open neighborhood of p in the identification topology on M, and if we take a single half-disc, its image in M is not a neighborhood of p and is not open, so π is not an open mapping.

Example 7 Consider the subspace $A = [0, 1] \cup (2, 3]$ of $\mathbb{R}$; it is a subspace of the space $X = [0, 1] \cup [2, 3]$ of Example 2. Suppose that we restrict the map p of Example 2 to A. Then

$$q = p|_A : A \to [0, 2]$$

is continuous and surjective, but it is not a quotient map since $(1, 2]$ is open in Y while $q^{-1}((1, 2]) = (2, 3]$ is closed in A. Note that $(1/2, 1]$ is open in A, $p|_A((1/2, 1]) = (1/2, 1]$ is not open in Y, so $p|_A : A \to Y$ is not an open map. Also note that $(2, 5/2]$ is closed in A, $p|_A((2, 5/2]) = (1, 3/2]$ is not closed in Y, so $p|_A : A \to Y$ is not a closed map.

So, if $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $p' : A \to p(A)$ obtained by restricting both the domain and range of p need not be a quotient map. However, it is easy to see that if A is open in X and p is an open map, then ($p' = p|_A$ is an open map and) p' is a quotient map; the same is true if both A and p are closed.

Example 8 Let Y be the subspace $(\overline{\mathbb{R}}_+ \times \mathbb{R}) \cup (\mathbb{R} \times 0)$ of $\mathbb{R} \times \mathbb{R}$; let $h = \pi_1|_Y$. For any subset U of $\mathbb{R}$, since

$$h^{-1}(U) \cap (\mathbb{R} \times 0) = U \times 0,$$

h is a quotient map while h is neither open nor closed. For instance, the set

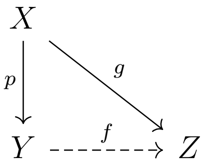
$$A = \{(x, y) \mid x^2 + (y - 2)^2 < 1,\ x \geq 0\} \text{ is open}$$

and the set

$$B = \{(x, y) \mid xy = 1,\ x > 0\} \text{ is closed in } Y,$$

while $h(A) = [0, 1)$ is not open and $h(B) = (0, \infty)$ is not closed in $\mathbb{R}$.

Theorem Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a continuous function that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a continuous function $f : Y \to Z$ such that $f \circ p = g$.



Proof For each $y \in Y$, since g is constant on $p^{-1}(\{y\})$, the set $g(p^{-1}(\{y\}))$ is a one-point set in Z and $f(y)$ can be defined as

$$f(y) = g(p^{-1}(\{y\})).$$

So, we have defined a map $f : Y \to Z$ such that

$$f(p(x)) = g(x) \quad \text{for each } x \in X = \bigcup_{y \in Y} \{p^{-1}(y)\}.$$

Given an open set V of Z, since g is continuous and p is a quotient map,

$$g^{-1}(V) = p^{-1}(f^{-1}(V))$$

is open in X and $f^{-1}(V)$ is open in Y. This shows that f is continuous.

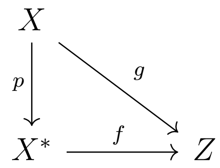
Theorem Let $g : X \to Z$ be a surjective continuous function. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\} = \bigcup_{z \in Z} g^{-1}(\{z\}).$$

Give X^* the quotient topology.

(a) If Z is Hausdorff, then so is X^*.

(b) The map g induces a bijective continuous function $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.



Proof By the preceding theorem, g induces a continuous function $f : X^* \to Z$; it is clear that f is bijective.

Suppose that f is a homeomorphism. Then both f and the projection $p : X \to X^*$ are quotient maps, so that $g = f \circ p$ is a quotient map.

Conversely, suppose that g is a quotient map. Given an open set V of X^*, since p is continuous,

$$g^{-1}(f(V)) = p^{-1}(V)$$

is open in X and $g : X \to Z$ is a quotient map, $f(V)$ is open in Z. Hence f maps open sets to open sets, so it is a homeomorphism.

If Z is Hausdorff, then given distinct points of X^*, their images under f are distinct and thus possess disjoint neighborhoods U and V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighborhoods of the two given points of X^*.